Heat transport by fluid flows with prescribed velocity fields

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We study the problem of heat transport by fluid flows with prescribed velocity fields. The advection-diffusion equation in two dimensions is solved for two velocity fields: (i) a circulation and (ii) a shear flow. These two flows focus separately on the two dominant features of the mean large-scale flow observed in turbulent convection experiments. We find that the Nusselt number, which measures the heat transport, scales respectively for the two velocity fields.

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I. INTRODUCTION

Rayleigh-Bénard convection in a fluid enclosed in a cell has been a system of much research interest. The dynamics is driven by an applied temperature difference across the height of the cell. The flow state is characterized by the geometry of the cell and two dimensionless numbers: the Rayleigh number $\mathrm{Ra}=\alpha g\Delta L^3/(\nu\kappa)$ and the Prandtl number $\mathrm{Pr}=\nu/\kappa$, where Δ is the applied temperature difference, L is the height of the cell, g the acceleration due to gravity, and α, ν, and κ are respectively the volume expansion coefficient, kinematic viscosity, and thermal diffusivity of the fluid. When Ra is sufficiently large, the convection becomes turbulent. In addition to the statistical properties of the temperature and velocity fluctuations, there is the central issue of heat transport by the fluid. The global heat transport is usually expressed as the dimensionless Nusselt number Nu, which is the ratio of the measured heat flux to the heat transported were there only conduction. Before the onset of convection, heat is transported only by conduction and Nu is identically equal to one. When convection occurs, heat is more effectively transported by the fluid due to its motion and Nu increases from 1.

The work of Libchaber and co-workers on turbulent convection in low temperature helium gas [1,2] showed that Nu has a simple power-law dependence on Ra:

$$\mathrm{Nu}\sim\mathrm{Ra}^{\beta} \tag{1}$$

and the exponent β is almost equal to 2/7, which is different from 1/3, the value that marginal stability arguments [3] would give. This result led to the development of several theories [2,4–6]. All these theories give $\beta=2/7$ but they were based on rather different physical assumptions. Moreover, some of the assumptions have not been supported by experiments, see, e.g., [7]. Further experimental investigations appeared to indicate a dependence of β on Pr, see, for example, Ref. [8] for a review of the experimental results. The situation is further complicated by two recent experimental results. First, Niemela *et al*. [10] made measurements in low temperature helium gas over a much larger range of Ra, from 10^6 to 10^{17}, and found β close to 0.31. Second, Xu *et al*. [11] studied turbulent convection in acetone in several experimental cells of different aspect ratios and concluded that there is no significant range of Ra over which the power-law behavior (1) holds. Furthermore, they showed that for $\mathrm{Ra}\geqslant 10^7$, the dependence of Nu on Ra can be represented by a combination of two power laws, which is consistent with a recent theory by Grossmann and Lohse [8,9]. Grossmann and Lohse's theory was based on a systematic decomposition of the thermal and kinetic energy dissipation into their bulk and boundary-layer contributions.

Another interesting feature observed in turbulent convection is the presence of a persistent large-scale mean flow that spans the whole experimental cell [12]. The maximum mean velocity of the flow was also found to scale as Ra to about 1/2 [13]. The presence of a large-scale flow naturally induces an interaction between the top and bottom thermal boundary layers. Such an interaction was taken to be absent in the marginal stability arguments. One obvious effect of the velocity field, which satisfies the no-slip boundary condition, is that it produces a shear near the boundaries, which was first studied in Ref. [5]. In convection, the velocity and temperature fields are coupled dynamically in a complicated fashion. On the one hand, the temperature field takes part in driving the flow and is so-called active. On the other hand, the resulting velocity field, in turn, shapes the temperature profile in a self-consistent manner.

In this paper, we study the simpler problem of heat transport by fluid flows with prescribed velocity fields. We study two cases, (i) a circulation and (ii) a shear flow, which focus separately on the two dominant features of the large-scale mean flow observed in turbulent convection, with an aim of gaining physical insights or understanding of heat transport in turbulent convection. The dependence of the heat transport on the Peclet number, as estimated by the parameters specifying the two flows, is investigated.

II. THE PROBLEM

We study the problem of heat transport by a fluid with a prescribed time-independent velocity field. To this end, we solve the steady state advection-diffusion equation

$$\vec{u}(x,y)\cdot\vec{\nabla}T(x,y)=\kappa\nabla^2T(x,y) \tag{2}$$

for a given incompressible velocity field $\vec{u}(x,y)$ in two dimensions: $0\leqslant x\leqslant L$ and $0\leqslant y\leqslant L$. A temperature difference of Δ is applied across the y-direction while no heat conduction is allowed across the x direction. That is, the temperature field $T(x,y)$ satisfies the following boundary conditions:

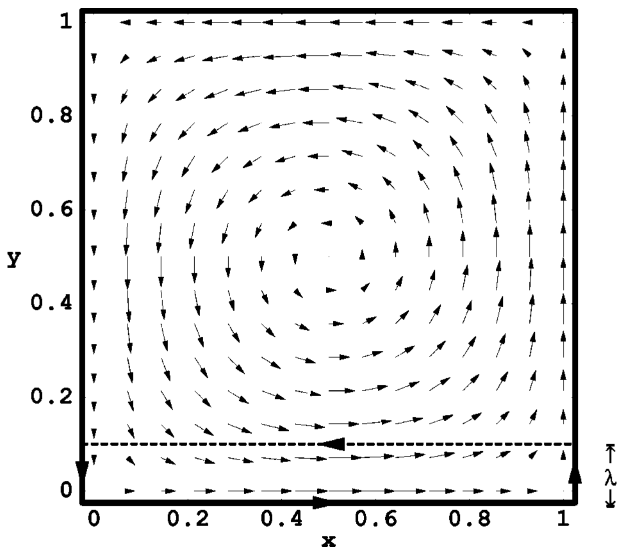

FIG. 1. The circulating flow given by Eq. (5). Both x and y are normalized by L. The size of the arrow indicates the relative magnitude of the velocity. The contour C that encloses the lower thermal boundary layer is also shown.

$$T(x,y=0)=\Delta; \qquad T(x,y=L)=0; \tag{3}$$

$$\frac{\partial T}{\partial x}(x=0,y)=\frac{\partial T}{\partial x}(x=L,y)=0. \tag{4}$$

The heat transport by the fluid is then calculated from the temperature field solved.

The mean large-scale flow observed in turbulent convection has two dominant features: it is a circulation that spans the whole experimental cell, and it produces a shear near the boundaries. To gain insights and understanding of how these two aspects of the velocity field affect or determine the heat transport, we study two different velocity fields. The first one is

$$u_x=u_0 \sin\left(\pi\frac{x}{L}\right)\cos\left(\pi\frac{y}{L}\right), \tag{5a}$$

$$u_y=-u_0 \cos\left(\pi\frac{x}{L}\right)\sin\left(\pi\frac{y}{L}\right), \tag{5b}$$

where u_0 is a constant. Equation (5) focuses on the circulating aspect of the mean large-scale flow. As seen from Fig. 1, the fluid undergoes a complete cycle across the whole two-dimensional space. The velocity vanishes at the center but u_x is finite along the boundaries $y=0$ and $y=L$ while u_y is finite along the boundaries $x=0$ and $x=L$. The velocity field, therefore, does not satisfy the no-slip boundary condition. Such a circulating flow has been studied in an investigation of the effects of a large-scale circulation on passive scalar statistics [14]. One of the interesting results found in that study is that the mean temperature profile is significantly altered by the velocity circulation, and develops two thermal boundary layers at $y=0$ and $y=L$. Each boundary layer is of thickness λ at the central axis of the cell and across which half the applied temperature difference drops. This mean temperature profile is just like that observed in turbulent convection. Thus the thermal boundary layers are not a consequence of the no-slip velocity boundary condition. Rather, their existence allows for an increase in the heat transport (Nu>1). In other words, the circulation enhances the heat transport. Our present interest is to study how the heat transport by the fluid depends on the strength of the circulation, as measured by the parameter u_0 or the dimensionless Peclet number (Pe), defined by u_0L/κ.

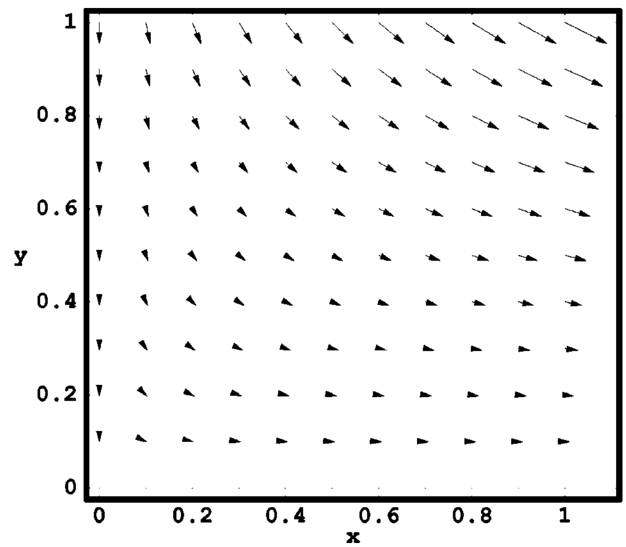

FIG. 2. Similar plot as in Fig. 1 for the shear flow given by Eq. (6) with a linear shear rate $\eta(x)=ax$.

The second velocity field studied is a shear along the boundary $y=0$, with a position-dependent shear rate $\eta(x)$:

$$u_x=\eta(x)y, \tag{6a}$$

$$u_y=-\frac{1}{2}\eta'(x)y^2. \tag{6b}$$

In Ref. [5], the large-scale flow in turbulent convection was approximated by a constant shear flow, represented by Eq. (6) with $\eta(x)=\eta$, near the bottom boundary and the effect of such a constant shear flow on the heat transport was studied. Here we consider a linear shear rate: $\eta(x)=ax$ with a being a constant, which is the simplest case that gives an exact analytical solution for the temperature field. Such a flow again does not satisfy the no-slip boundary condition on $x=0$, $x=L$, and $y=0$, as shown in Fig. 2. As in Ref. [5], this flow can be taken as an approximation of the large-scale flow near the bottom boundary in turbulent convection. Position-dependent shear rates have been considered by one of the present authors [15]. In this work, we study how the heat transport by the fluid depends on the parameter a or Pe defined by aL^3/κ. In this case, Pe gives a measure of the velocity as well as the shear rate of the flow.

For the circulation Eq. (5), we solve Eq. (2) numerically to get the temperature field $T(x,y)$. For the shear flow (6), we obtain an exact analytical solution for the temperature field. The heat transport by the fluid is the heat conducted across the boundary $y=0$. Thus, Nu is the ratio of the aver-

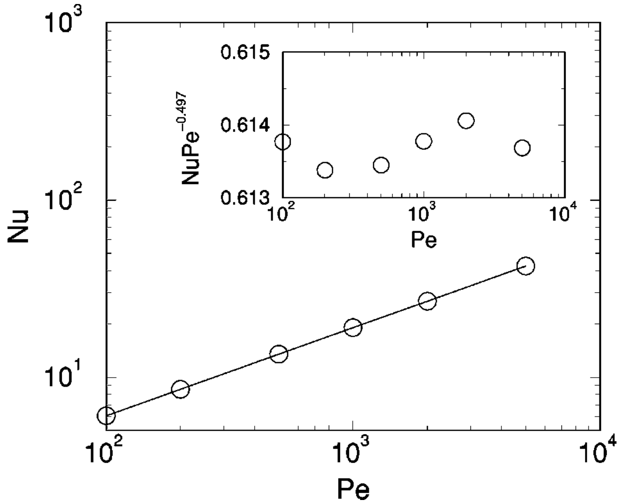

FIG. 3. Nu as a function of Pe for the circulating flow shown in Fig. 1. Here, Pe is taken as $u_0 L/\kappa$. The circles are numerical results while the solid line is the best linear fit with a slope of 0.497 in the log-log plot. In the inset, $\mathrm{NuPe}^{-0.497}$ is plotted versus Pe. The data points are constant within an error of 0.1%, showing the good quality of the power law.

age of the magnitude of the vertical temperature gradient over the boundary $y=0$ to Δ/L:

$$\mathrm{Nu}=\frac{\left\langle -\left.\frac{\partial T}{\partial y}\right|_{y=0}\right\rangle}{\frac{\Delta}{L}}. \tag{7}$$

Here, $\langle\cdots\rangle$ indicates an average over x, from $x=0$ to $x=L$.

III. RESULTS AND DISCUSSIONS

For the circulating flow Eq. (5), we numerically solve Eq. (2) and evaluate Nu. As expected, the heat transport is found to increase with the strength of the circulation. In Fig. 3, we show the dependence of Nu on Pe, which can be well represented by a power law with an exponent of a least-square-fit value of 0.497. In the inset of Fig. 3, $\mathrm{NuPe}^{-0.497}$ is plotted versus Pe. As can be seen, the data points are constant to within an error of 0.1%, showing the good quality of the power law.

We shall understand this result in the following. Since the velocity field is incompressible, Eq. (2) implies

$$\int_C (\vec{u}T-\kappa\vec{\nabla}T)\cdot\hat{n}\,dl=0 \tag{8}$$

over any closed curve C in the two-dimensional domain where $\hat{n}$ is the outward normal. In particular, choose the closed curve C, defined by $y=0$, $y=\lambda$, and $0\leqslant y\leqslant\lambda$ with $x=0$ and $x=L$, which encloses the thermal boundary layer at $y=0$ (see Fig. 1). Since u_x and $\partial T/\partial x$ vanish at $x=0$ and $x=L$, u_y vanishes at $y=0$ and $\partial T/\partial y\approx 0$ at $y=\lambda$, we have

$$\langle (u_y T)|_{y=\lambda}\rangle-\kappa\left\langle -\left.\frac{\partial T}{\partial y}\right|_{y=0}\right\rangle\approx 0. \tag{9}$$

Equation (9) states that the heat conducted across the lower boundary $y=0$ is approximately equal to the heat convected across the upper edge of the thermal boundary layer. Thus, we can estimate the heat transport by estimating $\langle (u_y T)|_{y=\lambda}\rangle$. As $\lambda/L\ll 1$, $u_y(x,y=\lambda)\approx -u_0\pi(\lambda/L)\cos(\pi x/L)$. A horizontal temperature difference is induced by the circulation with the boundary $x=L$ being hotter than the boundary $x=0$. We estimate $T(x,y=\lambda)$ to have a similar cosine dependence on x as $(\Delta/2)[1-(8k^2/\pi)\cos(\pi x/L)]$, where k is some dimensionless constant. Hence we have

$$k^2\left(\frac{u_0 L}{\kappa}\right)\left(\frac{2\lambda}{L}\right)\approx\frac{\left\langle -\left.\frac{\partial T}{\partial y}\right|_{y=0}\right\rangle}{\frac{\Delta}{L}}. \tag{10}$$

Using Eq. (7) and the estimate of Nu by $L/(2\lambda)$, Eq. (10) then gives

$$\mathrm{Nu}\approx k\,\mathrm{Pe}^{1/2}. \tag{11}$$

Our theoretically estimated exponent 1/2 agrees well with the numerical result of 0.497.

For the shear flow, we obtain an exact analytical solution of the temperature profile. Let $T(x,y)$ be separable in x and y, that is,

$$T(x,y)=F(x)G(y). \tag{12}$$

Substituting Eq. (12) into Eq. (2) and rearranging terms give

$$axy\frac{F'(x)}{F(x)}-\kappa\frac{F''(x)}{F(x)}=\kappa\frac{G''(y)}{G(y)}+\frac{ay^2}{2}\frac{G'(y)}{G(y)}, \tag{13}$$

where a prime denotes a derivative with respect to the corresponding variable: x or y. The right-hand side of Eq. (13) is a function of y only when the left-hand side is a function of both x and y. This can only be the case if $F(x)$ is a constant, say c, and Eq. (13) becomes

$$\kappa\frac{G''(y)}{G(y)}+\frac{1}{2}ay^2\frac{G'(y)}{G(y)}=0. \tag{14}$$

The exact analytical solution for T is thus

$$T(y)=cG(y)=\Delta\left[c_1+c_2\int_0^{y/L}\exp\left(-\frac{aL^3}{6\kappa}q^3\right)dq\right] \tag{15}$$

with the constants c_1 and c_2 fixed by the two boundary conditions in Eq. (3):

$$c_1=1, \tag{16}$$

$$c_2=-3\left(\frac{aL^3}{6\kappa}\right)^{1/3}\left[\gamma\left(\frac{1}{3},\frac{aL^3}{6\kappa}\right)\right]^{-1}, \tag{17}$$

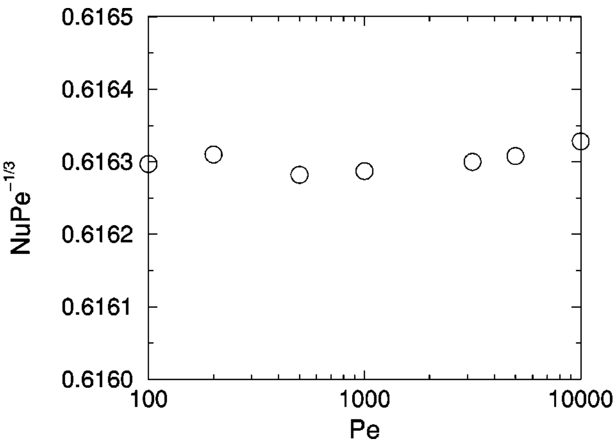

FIG. 4. $\mathrm{NuPe}^{-1/3}$ vs Pe for the shear flow Eq. (6) with a linear shear rate $\eta(x)=ax$. Here, Pe is taken as aL^3/κ. The data points agree with the value of 0.6163 to within an error of less than 0.005%.

where γ is the incomplete Gamma function [16]. The other two boundary conditions in Eq. (4) are automatically satisfied.

When $\mathrm{Pe}=aL^3/\kappa$ is large,

$$\gamma\left(\frac{1}{3},\frac{aL^3}{6\kappa}\right)\approx\Gamma(1/3)=2.6789 \;\; [\text{Ref. } [16]],$$

we have,

$$T(y)\approx\Delta\left[1-0.6163\left(\frac{aL^3}{\kappa}\right)^{1/3}\int_0^{y/L}\exp\left(-\frac{aL^3}{6\kappa}q^3\right)dq\right]. \tag{18}$$

Using Eq. (7), we find

$$\mathrm{Nu}=0.6163\,\mathrm{Pe}^{1/3} \tag{19}$$

for large Pe. In Fig. 4, we show $\mathrm{NuPe}^{-1/3}$ vs Pe using numerically evaluated results of Nu. The data points agree with the value of 0.6163 to within an error of less than 0.005%. Thus, for the shear flow with a linear shear rate, Nu scales with Pe or equivalently the normalized shear rate to an exponent 1/3.

IV. SUMMARY AND CONCLUSIONS

In this paper, we have studied the problem of heat transport by fluid flows with prescribed velocity fields. This problem is of interest in engineering applications. Our objective is, however, to gain insights into the more difficult problem of heat transport in turbulent convection. The two velocity fields that we have studied are, therefore, chosen such that they focus separately on the two dominant features, namely, the circulating and the shear-generating aspects of the mean large-scale flow observed in turbulent convection.

We have solved numerically and analytically the heat transport respectively for a circulating and a shear flow with linear shear rate. As expected, the heat transport is found to increase with the Peclet number. In both cases, we have found that Nu scales as Pe to some exponent, and the value of the exponent is respectively 1/2 and 1/3 for the two velocity fields. For the shear flow with linear shear rate, this result is also equivalent to a scaling between Nu and the normalized shear rate with the same exponent 1/3. We have shown that our numerical result for the circulating flow can be understood by some theoretical estimates of $u_y T$ along the edge of the thermal boundary layer.

As discussed in the Introduction, the mean large-scale flow observed in turbulent convection has both the characteristics of the two velocity fields studied here: it is a circulation as well as approximately a shear within a viscous boundary layer near the boundaries of the cell. Our study thus suggests that Nu would generally depend on both Pe, defined by the maximum average velocity of the flow and the normalized shear rate. It would, therefore, be interesting to study the heat transport by a prescribed velocity field that has both features. We shall report such a study and the results elsewhere [17].

There are two limits in which one expects the effect of the mean large-scale flow on heat transport to be dominated by one of the two characteristics. The first limit is when the thermal boundary layer is much thinner than the viscous boundary layer, which should be the case when Pr is very large. In this limit, the velocity field can be well approximated as a simple shear within the thermal boundary layer. Our result would suggest a scaling of Nu with the normalized shear rate to 1/3. This scaling result was obtained earlier in Refs. [5,15] for turbulent convection. The second limit is when the viscous boundary layer, is much thinner than the thermal boundary layer, which should be the case when Pr is very small. In this limit, the velocity field within the thermal boundary layer might be taken as a circulation plus a thin shear layer. If the ratio of the thickness of the thermal boundary layer to that of the viscous boundary layer is large enough such that the contribution of the thin shear layer to the heat transport can be neglected, our result would then suggest a scaling of Nu with Pe, defined by the maximum average velocity, to 1/2. Together with the observation that the maximum mean velocity of the large-scale flow scales with Ra to an exponent of about 1/2 [13], our result would further suggest a scaling of Nu with Ra to 1/4. Such a scaling was observed in several experimental studies [18,19] and a numerical study [20] of turbulent convection in fluids with small Pr. This scaling result was also obtained in Refs. [8,20] by dimensional arguments. A key step in these dimensional arguments is to approximate $\partial^2T/\partial y^2$ by $\Delta/(2\lambda^2)$ within the thermal boundary layer. This approximation is, however, not obvious as the temperature should be quite well represented by a linear function in y within the thermal boundary layer.

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